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Coherent States in the Form of a Quantum Group

Lei Ma,[1] Zhong Tang,[1] and Yong-de Zhang[1]

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A systematic approach is proposed to give coherent states in the form of a quantum group. We construct some after considering the problem of positive measure.

1. INTRODUCTION

Since Glauber put forward the concept of coherent state 30 years ago, it has been studied and used in many realms of physics. According to Klauder and Skagerstam (1985), the two common properties of all coherent states are:

(i) *Continuity*. The coherent state $|\xi\rangle$ is a strong continuous function for parameter ξ.
(ii) *Supercompleteness*. There exists a positive measure $\delta\xi$ in the parameter space to make the following unit decomposition tenable:

$$I=\int |\xi\rangle\langle\xi|\,\delta\xi \tag{1.1}$$

The above integral extends all over the parameter space.

Recently, much attention has been paid to quantum groups, or q-analogue algebras. Sun and Ge (1991) gave a boson Fock representation in a quantum group. Bracken *et al.* (1991) sought a coherent state in the form of a quantum group. In this paper we construct some coherent states in quantum group theory. We first consider the problem of positive measure, then construct some coherent states, and finally discuss the spin coherent state under the q-analogue.

[1]Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China.

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For any Ω which is a c-number or an operator in Fock space, we define $[\Omega]$ as follows:

$$[\Omega]=\frac{q^{\Omega}-q^{\Omega}}{q-q^{-1}} \tag{1.2}$$

where we let q be a positive real number. Then the operators $\tilde{a}$, $\tilde{a}^{+}$ are defined as follows:

$$\begin{aligned}\tilde{a}&=a\left(\frac{[N]}{N}\right)^{1/2}=\left(\frac{[N+1]}{N+1}\right)^{1/2}a\\ \tilde{a}^{+}&=a^{+}\left(\frac{[N+1]}{N+1}\right)^{1/2}=\left(\frac{[N]}{N}\right)a^{+}\end{aligned} \tag{1.3}$$

Here a and a^{+} are the canonical annihilation operator and creation operator, which satisfy $[a, a^{+}]=aa^{+}-a^{+}a=1$; $\tilde{a}$, $\tilde{a}^{+}$, and the operator $N=a^{+}\, a$ obey the commutation relations

$$\begin{aligned}[][N, \tilde{a}^{+}]&=\tilde{a}^{+}\\ [\tilde{a}, N]&=\tilde{a}\\ \tilde{a}\tilde{a}^{+}-q\tilde{a}^{+}\tilde{a}&=q^{-N}\end{aligned} \tag{1.4}$$

For the Fock states in a quantum group, we define

$$\begin{aligned}&\tilde{a}|\tilde{0}\rangle=0\\ &N|\tilde{m}\rangle=m|\tilde{m}\rangle\\ &\tilde{a}^{+}|\tilde{m}\rangle=([m+1])^{1/2}|\widetilde{m+1}\rangle\\ &\tilde{a}|\tilde{m}\rangle=([m])^{1/2}|\widetilde{m-1}\rangle, m=0, 1, 2, \ldots\end{aligned} \tag{1.5}$$

We find that

$$|\tilde{m}\rangle=\frac{1}{([m]!)^{1/2}}(\tilde{a}^{+})^{m}|\tilde{0}\rangle \tag{1.6}$$

where

$$\begin{aligned}&[m]!=[m]\cdot[m-1]\cdot\,\cdots\,\cdot[1], \qquad m=1, 2, 3, \ldots\\ &[0]!=1\end{aligned}$$

2. SOME CHARACTERISTICS OF THE POSITIVE MEASURE

In general, we suppose that a coherent state in the form of a quantum group can be expanded in Fock space as follows:

$$|\xi\rangle=A(\xi)\sum_{m=0}^{\infty}C_m(\xi)|\tilde{m}\rangle \tag{2.1}$$

and satisfies

$$1=\int |\tilde{\xi}\rangle\langle\tilde{\xi}| g(\xi) d^2\xi \tag{2.2}$$

where ξ is a complex parameter number, the functions $C_m(\xi)$ of ξ are nonzero coefficients, $A(\xi)$ is a normalization coefficient,

$$A(\xi)=\left(\sum_{m=0}^{\infty} |C_m(\xi)|^2\right)^{-1/2} \tag{2.3}$$

and $g(\xi)$ is a positive function, which we call the positive measure of $|\tilde{\xi}\rangle$. For convenience, we denote $g(\xi)A^2(\xi)\equiv G(\xi)$. Since $A(\xi)$ can be obtained when the $C_m(\xi)$ are known, $G(\xi)$ is directly dependent on $g(\xi)$. Equation (2.1) reduces to

$$|\tilde{\xi}\rangle = A(\xi) f(\xi, \tilde{a}^+)|\tilde{0}\rangle \tag{2.4}$$

here

$$f(\xi, \tilde{a}^+)=\sum_{m=0}^{\infty} \frac{C_m(\xi)}{\sqrt{[m]!}} (\tilde{a}^+)^m \tag{2.5}$$

From the completeness, we get

$$1=\int |\tilde{\xi}\rangle\langle\tilde{\xi}| g(\xi)\, d^2\xi = \sum_{m,n=0}^{\infty} |\tilde{m}\rangle\langle\tilde{n}| \int G(\xi) C_m(\xi) C_n^*(\xi)\, d^2\xi$$

Then we obtain

$$\int G(\xi) C_m(\xi) C_n^*(\xi)\, d^2\xi = \delta_{mn}, \qquad m, n=0, 1, 2, \ldots \tag{2.6}$$

These equations are one of the characteristics of $g(\xi)$.

In addition, if we let $\tilde{a}^+ \to \eta$, $\tilde{a} \to \eta^*$ (η is an arbitrary complex constant), we obtain

$$\begin{aligned}\int G(\xi)\, |f(\xi, \eta)|^2\, d^2\xi &= \int G(\xi) f(\xi, \eta) f^*(\xi, \eta)\, d^2\xi \\ &= \sum_{m,n=0}^{\infty} \frac{\eta^m \eta^{*n}}{([m]!\, [n]!)^{1/2}} \int G(\xi) C_m(\xi) C_n^*(\xi)\, d^2\xi \\ &= \sum_{m=0}^{\infty} \frac{|\eta|^{2m}}{[m]!} \end{aligned} \tag{2.7}$$

We denote

$$\exp_q(\Omega) \equiv \sum_{m=0}^{\infty} \frac{\Omega^m}{[m]!} \tag{2.8}$$

where Ω is a number or an operator. Then equation (2.7) changes into

$$\int G(\xi)\,|f(\xi, \eta)|^2\, d^2\xi = \exp_q(|\eta|^2) \tag{2.9}$$

Equation (2.9) is another characteristic of the positive measure $q(\xi)$.

3. CONSTRUCTION OF COHERENT STATES IN THE FORM OF A QUANTUM GROUP

Now we construct coherent states in the form of a quantum group. First we get states and then find the positive measures that make them complete.

We first consider states with the following form:

$$|\tilde{\xi}\rangle = A(\xi) \sum_{m=0}^{\infty} \lambda_m \xi^m |\tilde{m}\rangle \tag{3.1}$$

where $\{\lambda_m\}$ is a set of nonzero coefficients that do not contain the parameter ξ, and $A(\xi) = (\sum_{m=0}^{\infty} |\lambda_m|^2 |\xi|^{2m})^{-1/2}$. If a coherent state has the above form, there exists a function $g(\xi)$ that is the positive measure of $|\tilde{\xi}\rangle$. From equation (2.6), we get

$$\int G(\xi)\xi^m \xi^{*n}\, d^2\xi = \frac{\delta_{mn}}{\lambda_m \lambda_n^*}, \qquad m, n = 0, 1, 2, \ldots \tag{3.2}$$

We choose $G(\xi)$ to be a function of $|\xi|^2$ and denote $G(\xi) \equiv G_1(|\xi|^2)$. Then

$$\int_0^{\infty} G_1(t) t^m\, dt = \frac{1}{\pi |\lambda_m|^2}, \qquad m = 0, 1, 2, \ldots \tag{3.3}$$

From the Laplace transformation formula, we find

$$F(s) \equiv \int_0^{\infty} G_1(t) e^{-st}\, dt = \sum_{m=0}^{\infty} \frac{(-s)^m}{m!} \int_0^{\infty} G_1(t) t^m\, dt = \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-s)^m}{m! |\lambda_m|^2} \tag{3.4}$$

Hence

$$G_1(|\xi|^2) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \exp(s|\xi|^2) F(s)\, ds \tag{3.5}$$

where σ is a positive number. In the following we give two examples.

(i) We consider the case of $\lambda_m = 1/([m]!)^{1/2}$. The relevant coherent state $|\tilde{\xi}\rangle$ is

$$|\tilde{\xi}\rangle = A(\xi)\exp_q(\xi\tilde{a}^+)|\tilde{0}\rangle \tag{3.6}$$

where

$$A(\xi) = \{\exp_q(|\xi|^2)\}^{-1/2} \tag{3.7}$$

The above state is an eigenstate of $\tilde{a}$,

$$\tilde{a}|\tilde{\xi}\rangle = \xi|\tilde{\xi}\rangle \tag{3.8}$$

Inserting $\lambda_m = 1/([m]!)^{1/2}$ into equation (3.4), we have

$$F(s) = \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{[m]!}{m!}(-s)^m \tag{3.9}$$

In the above state, if $q \to 1$, so that $F(s) = 1/(1+s)$, we get that

$$G_1(|\xi|^2)|_{q\to 1} = \frac{1}{\pi}\exp(|\xi|^2), \quad \text{i.e.,} \quad g(\xi)|_{q\to 1} = \frac{1}{\pi} \tag{3.10}$$

and we come back to the usual canonical coherent state.

We can also construct the following coherent state $[\lambda_n = 1/(n!)^{1/2}]$:

$$|\tilde{\xi}\rangle = A(\xi)\sum_{n=0}^{\infty}\frac{\xi^n}{(n!)^{1/2}}|\tilde{n}\rangle = A(\xi)\sum_{n=0}^{\infty}\frac{\xi^n(\tilde{a}^+)^n}{(n![n]!)^{1/2}}|\tilde{0}\rangle \tag{3.11}$$

where $A(\xi) = \exp(-\frac{1}{2}|\xi|^2)$. Whether $q=1$ or not, the positive measure is always

$$g(\xi) = \frac{1}{\pi} \tag{3.12}$$

Actually, the above state is just the usual canonical coherent state.

(ii) We can construct a coherent state with the hypergeometric function. An example is

$$|\tilde{\xi}\rangle = A(\xi)\sum_{m=0}^{\infty}\left(\frac{(a)_m}{m![m]!(\beta)_m}\right)^{1/2}(\xi\tilde{a}^+)^m|\tilde{0}\rangle \tag{3.13}$$

$$\mathrm{A}(\xi) = [{}_1F_1(\alpha;\beta;|\xi|^2)]^{-1/2} \tag{3.14}$$

where α and β are two positive real numbers. ${}_1F_1(\alpha;\beta;\eta)$ is a ${}_1F_1$-type hypergeometric function of η, namely

$$ {}_1F_1(\alpha;\beta;\eta)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!(\beta)_n}\eta^n \tag{3.15}$$

From equation (3.4) we can easily get

$$F(s)=\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(\beta)_m}{(\alpha)_m}(-s)^m=\frac{1}{\pi}{}_2F_1(1,\beta;\alpha;-s) \tag{3.16}$$

${}_2F_1(1,\beta;\alpha;-s)$ is a ${}_2F_1$-type hypergeometric function. Thus we find

$$G_1(|\xi|^2)=\frac{1}{2\pi^2 i}\int_{\sigma-i\infty}^{\sigma+i\infty}\exp(s|\xi|^2){}_2F_1(1,\beta;\alpha;-s)\,ds \tag{3.17}$$

Besides the above states with the form of equation (3.1), we can also construct other coherent states. For instance, we can make a coherent state with Hermite polynomial $H_n(\xi)$ as follows:

$$|\tilde{\xi}\rangle=A(\xi)\sum_{n=0}^{\infty}\frac{H_n(\xi)}{(n!)^{1/2}}\alpha^n|\tilde{n}\rangle \tag{3.18}$$

where α is a complex constant number and must satisfy the relation $0<|\alpha|<\sqrt{2}/2$. From the identity (Wang and Guo, 1979)

$$\sum_{n=0}^{\infty}\frac{(t/2)^n}{n!}H_n(x)H_n(y)=(1-t^2)^{-1/2}\exp\left\{\frac{2xyt-(x^2+y^2)t}{1-t^2}\right\},\qquad -1<t<1 \tag{3.19}$$

we easily find

$$A(\xi)=(1-4|\alpha|^4)^{1/4}\exp\left\{\frac{2|\alpha|^2[|\alpha|^2(\xi^2+\xi^{*2})-|\xi|^2]}{1-4|\alpha|^4}\right\} \tag{3.20}$$

and we get a positive measure for this state as follows:

$$g(\xi)=\frac{4|\alpha|^2}{\pi(1-4|\alpha|^4)} \tag{3.21}$$

Proof. Since the state of equation (3.18) can be reduced to

$$|\tilde{\xi}\rangle=A(\xi)\sum_{m=0}^{\infty}\frac{H_m(\xi)}{m!}(\alpha a^+)^m|0\rangle \tag{3.22}$$

according to Wang and Guo (1979), we obtain

$$|\tilde{\xi}\rangle = A(\xi)\exp(-\alpha^2 a^{+2} + 2\xi\alpha a^+)|0\rangle \tag{3.23}$$

Let $\mathrm{Re}(\xi) = x$, $\mathrm{Im}(\xi) = y$; we find

$$\int |\tilde{\xi}\rangle\langle\tilde{\xi}|\, d^2\xi$$

$$= :\int (1-4|\alpha|^4)^{1/2} \exp\left\{\frac{4|\alpha|^2[(\xi^2+\xi^{*2})|\alpha|^2 - |\xi|^2]}{1-4|\alpha|^4}\right.$$

$$\left. -\alpha^2 a^{+2} + 2\xi\alpha a^+ - \alpha^{*2}a^2 + 2\xi^*\alpha^* a - a^+ a\right\} d^2\xi:$$

$$= (1-4|\alpha|^4)^{1/2} :\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp\left\{-4|\alpha|^2\left(\frac{x^2}{1+2|\alpha|^2} + \frac{y^2}{1-2|\alpha|^2}\right)\right.$$

$$+ 2(\alpha a^+ + \alpha^* a)x + 2i(\alpha a^+ - \alpha^* a)y$$

$$\left. - \frac{1+2|\alpha|^2}{4|\alpha|^2}(\alpha a^+ + \alpha^* a)^2 + \frac{1-2|\alpha|^2}{4|\alpha|^2}(\alpha a^+ - \alpha^* a)^2\right\}:$$

$$= (1-4|\alpha|^4)^{1/2} :\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy$$

$$\times \exp\left\{-4|\alpha|^2\left[\frac{x}{(1+2|\alpha|^2)^{1/2}} - \frac{(1+2|\alpha|^2)^{1/2}}{4|\alpha|^2}(\alpha a^+ + \alpha^* a)\right]^2\right.$$

$$\left. -4|\alpha|^2\left[\frac{y}{(1-2|\alpha|^2)^{1/2}} - \frac{i(1-2|\alpha|^2)^{1/2}}{4|\alpha|^2}(\alpha a^+ - \alpha^* a)\right]^2\right\}:$$

$$= \frac{\pi(1-4|\alpha|^4)}{4|\alpha|^2} \tag{3.24}$$

Hence, the measure function of $|\tilde{\xi}\rangle$ is not dependent on ξ and is given by

$$g(\xi) = \frac{4|\alpha|^2}{\pi(1-4|\alpha|^4)}$$

4. THE SPIN COHERENT STATE IN THE FORM OF A QUANTUM GROUP

Finally, we discuss the spin coherent state simply in the form of a q-analogue. The angular momentum operators $\tilde{J}_+$, $\tilde{J}_-$, $\tilde{J}_0$ in the quantum group are defined as follows:

$$\begin{aligned}\tilde{J}_+ &= \tilde{a}_1^+ \tilde{a}_2 \\ \tilde{J}_- &= \tilde{a}_1 \tilde{a}_2^+ \\ \tilde{J}_0 &= \tfrac{1}{2}(N_1 - N_2)\end{aligned} \tag{4.1}$$

The commutation relation between $\tilde{J}_+$ and $\tilde{J}_-$ is

$$[\tilde{J}_+, \tilde{J}_-] = [2J_0] \tag{4.2}$$

The states $\{|\widetilde{j, m}\rangle\}$ are a two-dimensional complete basis set,

$$|\widetilde{j, m}\rangle = |\widetilde{j+m}\rangle \otimes |\widetilde{j-m}\rangle, \qquad m = -j, -j+1, \ldots, j \tag{4.3}$$

We write

$$\binom{m}{n}_{(q)} = \frac{[m]!}{[n]![m-n]!}, \qquad m \geq n \geq 0 \tag{4.4}$$

Then the usual spin-coherent state $|\xi\rangle = A(\xi)\exp(\xi J_-)|j, j\rangle$ in Klauder and Skagerstam (1985) can be reduced to the following form with q-analogue operators:

$$|\xi\rangle = A(\xi) \sum_{m=0}^{2j} \frac{1}{[m]!} \left\{ \frac{\binom{2j}{m}}{\binom{2j}{m}_{(q)}} \right\}^{1/2} (\xi \tilde{J}_-)^m |j, j\rangle \tag{4.5}$$

Similarly, from the way the usual spin state is constructed in Klauder and Skagerstam (1985), we construct the spin-coherent state in the form of a quantum group as follows:

$$|\tilde{\xi}\rangle = A(\xi)\exp_q(\xi \tilde{J}_-)|\widetilde{j, j}\rangle \tag{4.6}$$

Let us write

$$(1+\eta)^m_{(q)} = \sum_{n=0}^{m} \binom{m}{n}_{(q)} \eta^n \tag{4.7}$$

Then we get

$$A(\xi) = \{(1 + |\xi|^2)^{2j}_{(q)}\}^{-1/2} \tag{4.8}$$

Since we often need to construct coherent states in practical problems, this work extends the uses of quantum groups.

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